2.4 Haar measure and homogeneous spaces

In this section we are going to focus on the existence and uniqueness for the so-called Haar measure on locally compact Hausdorff top. groups and on Weil's criterion for the existence of invariant measures on homogeneous spaces of locally compact Hausdorff groups.

The key statement that we would like to generalize is the following.

Theorem: the Lebesgue measure λ is the unique complete translation-invariant measure on a σ-algebra containing the Borel sets such that $\lambda([0,1]^n) = 1$.

The main theorem that we are going to discuss below will imply the one stated above when applied to $G = (\mathbb{R}^n, +)$.

2.5.1 Haar measure.

$G \times X \longrightarrow X.$
$(g, x) \longmapsto gx.$
· $ex = x \quad \forall x \in X.$
· $g_1(g_2 x) = (g_1 g_2)x$

Terminology

A left action of a top. group G on a top. space X is said to be continuous if the action map $G \times X \longrightarrow X$ is continuous.

Given any map $F : X \longrightarrow Y$ we set:
$$(\lambda(g) F)(x) := F(g^{-1}x).$$

When G and X are locally compact and Hausdorff denoting by $C_c(X)$ the space of compactly supported continuous functions then: $\forall g \in G$

$$\lambda(g) : C_c(X) \longrightarrow C_c(X)$$

is an endomorphism and $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$

We are going to rely on the following classical result in measure theory.

(see [Rudin "Real and complex analysis" Thm 2.14]).

Theorem: [Riesz Representation Theorem] 2.39.

Let X be locally compact and Hausdorff and let $\Lambda : C_c(X) \longrightarrow \mathbb{C}$ be a positive linear functional. Then there are a σ-algebra $\mathfrak{M}$ containing all Borel subsets of X and a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that:

a) $\Lambda(f) = \int_X f(x)\, d\mu(x), \quad \forall f \in C_c(X)$

with the additional properties

b) $\mu(K) < \infty \quad \forall K \subset X$ compact.

c) $\forall E \in \mathfrak{M}.\ \mu(E) = \inf\{\mu(V) : E \subseteq V,\ V \text{ open}\}$

d) $\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}$ holds $\forall E \subseteq X$ open and $\forall E \in \mathfrak{M}$.

s.t. $\mu(E) < \infty$

e) If $E \in \mathfrak{M}$, $A \subseteq E$ and $\mu(E) = 0$
then $A \in \mathfrak{M}$.

$\boxed{\text{Note:}}$ Positive means that if $f \in C_0(X)$
has values in $[0, +\infty)$ then $\Lambda(f) \geq 0$.

If now $G \times X \to X$ is a continuous left action and $\Lambda \in C_c(X)^*$ is a linear functional, we can define

$$(\lambda^*(g)\,\Lambda)(f) := \Lambda(\lambda(g)^{-1} f) \quad \text{and}$$

obtain an endomorphism

$$\lambda^*(g) : C_c(X)^* \longrightarrow C_c(X)^*$$

with $\lambda^*(g_1 g_2) = \lambda^*(g_1)\,\lambda^*(g_2)$

$\boxed{\text{Exercise 2.40}}$

If Λ is a positive linear functional and μ denotes the corresponding regular

Borel measure obtained by Thm 2.39

then $\lambda^*(g)\Lambda$ is represented by the

measure $g_* \mu$ where

Push-forward measure

$$g_*\mu(A) := \mu(g^{-1}A) \quad \forall A \in \mathfrak{M}.$$

$\boxed{\text{Definition 2.41}}$

A <u>left (invariant) Haar functional</u> on a locally compact Hausdorff group G is a <u>positive</u> <u>non-zero</u> <u>functional</u>

$$\Lambda : C_c(G) \longrightarrow \mathbb{C}$$

such that $\lambda^*(g)\Lambda = \Lambda \quad \forall g \in G$

Via Thm 2.39

The associated regular Borel measure μ is called a <u>left (invariant) Haar measure</u>.

G acts on itself by multiplication

One can define analogously <u>right (invariant) Haar functionals</u> and <u>right (invariant) Haar measures</u>.

⇝ See below

The main theorem we will focus on is

Theorem 2.42 [Haar 1933]

Let G be a locally compact Hausdorff group. Then there exists a left invariant Haar measure and it is unique up to multiplication by an element of $\mathbb{R}_{>0}$, i.e., up to a scalar multiple in $\mathbb{R}_{>0}$, there exists a unique positive regular Borel measure μ on G such that for every measurable set $E \subseteq G$ and all $g \in G$ it holds

$$\mu(gE) = \mu(E).$$

= As in the statement of Thm 2.39.

We are going to prove only the uniqueness part of the statement.

Let's start with some preliminary observations.

Exercise. 2.43

For $f \in C_c(G)$ and $g \in G$ we let:

$(\rho(g) f)(x) := f(xg)$

Then: $\rho(g) \in \mathrm{End}(C_c(G))$ and

$\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$

Note: If $\Lambda \in C_c(G)^*$ is a linear functional. we define.

$$(\rho^*(g)\Lambda)(f) := \Lambda(\rho(g) f)$$

Definition:

Λ is a <u>right invariant Haar functional</u> if it is positive and $\rho^*(g)\Lambda = \Lambda$. $\forall g \in G$.

The corresponding measure is called a <u>right (invariant) Haar measure.</u>

Lemma 2.44

Let $\check{f}(x) := f(x^{-1})$. If Λ is a left Haar functional. then $\Lambda'(f) := \Lambda(\check{f})$. defines a right

Haar functional.

Corollary 2.45

Thm 2.42 holds for right Haar functionals and right Haar measures as well.

Proof of Lemma 2.44.

We let μ be the measure representing Λ, i.e.,

$$\Lambda(f) = \int_G f(x)\,d\mu(x).$$

We need to verify that $\Lambda'(\rho(g)f) = \Lambda'(f)$ $\forall g \in G$ and $\forall f \in C_c(G)$.

Note that:

$$(\check{\rho(g)f})(x) = (\rho(g)f)(x^{-1}) \qquad (*)$$
$$= f(x^{-1}g)$$

Hence $\Lambda'(\rho(g)f) \overset{\text{def.}}{=} \Lambda((\rho(g)f)\check{\ })$

$$= \int_G f(x^{-1}g)\,d\mu(x)$$

By $(*)$

$$= \int_G \check{f}(g^{-1}x)\, d\mu(x)$$

$$= \int_G \check{f}(x)\, d\mu(x)$$

By left-invariance of μ.

$$= \Lambda(\check{f}) \overset{\text{def.}}{=} \Lambda'(f). \quad \square$$

Later we will be applying Fubini's theorem. The application can be justified. If you are used to considering only σ-finite spaces note the following:

Lemma 2.46

If G is locally compact, Hausdorff, connected, and μ is a Haar measure on G then μ is σ-finite.

$G = \cup G_n$, $\mu(G_n) < \infty$

Proof.

By Lemma 2.32 (i) we can find an open neigh. V of e with $V = V^{-1}$.

Since G is loc. compact Hausdorff we can also assume that V has compact closure. $\rightsquigarrow$ $V \subseteq K$ with K compact wlog, then $\bar{V}$ is compact.

By Prop. 2.31 (iv) we have: $\bigcup_{n \geq 1} V^n = G$.

Since V^n is open with compact closure for each n, $\mu(V^n) < \infty$. Hence μ is σ-finite on G $\square$

$\boxed{\text{Lemma } 2.47}$

Let G be locally compact and Hausdorff with left Haar measure μ. Then:

(i) $\operatorname{supp}(\mu) = G$.

(ii) If $h \in C(G)$ is such that

$$\int_G h(x)\, \varphi(x)\, d\mu(x) = 0$$

$\forall\, \varphi \in C_c(G)$ then $h \equiv 0$.

Proof

(i) Recall that $\operatorname{supp} \mu := \{ x \in G :$

for every $U \ni x$ open $\mu(U) > 0\}$.

Since $\mu \neq 0$. there exists $f \in C_c(G)$ s.t. $\int f \, d\mu \neq 0$. We assume wlog. that $\int f \, d\mu > 0$.

Let $K := \operatorname{supp} f$. and note $\mu(K) > 0$.

otherwise $\int f d\mu = 0$

If $G \neq \operatorname{supp} \mu$, then there exists $x \in G \setminus \operatorname{supp}\mu$. and $U \ni x$ open with $\mu(U) = 0$.

Claim: there exist $g_1, \dots, g_n \in G$ s.t. $K \subseteq g_1 U \cup \dots \cup g_n U$ (Exercise.)

By the claim.

$$\begin{aligned}\mu(K) &\leq \mu(g_1 U \cup \dots \cup g_n U) \\ &\leq \mu(g_1 U) + \dots + \mu(g_n U) \\ &= \mu(U) + \dots + \mu(U) = 0.\end{aligned}$$

μ is left invariant

$\mu(U) = 0$

contradiction $\square$

ii). We will show that $h(e)=0$. The proof that $h(g)=0$ $\forall g \in G$ is analogous.

Let $\varepsilon > 0$. By continuity there is an open neigh $V \ni e$ s.t. $\forall g \in V$:

$$|h(g) - h(e)| < \varepsilon.$$

By <u>Urysohn's Lemma</u> there exists $p \in C_c(G)$ s.t. $p \geq 0$, $p(e) > 0$ and $\operatorname{supp} p \subseteq V$.

See [Rudin, Real and complex analysis, Lemma 2.12]

$\leadsto$ this is weaker than the usual Urysohn's Lemma for normal spaces.

Since $\int_G h(g) p(g)\, d\mu(g) = 0$ $\forall p \in C_c(G)$

we have:

$$|h(e)| \left| \int_G p(g)\, d\mu(g) \right|$$

$$= \left| \int_G h(e) p(g)\, d\mu(g) \right|$$

$$= \left| \int_G h(g) p(g)\, d\mu(g) - \int_G h(e) p(g)\, d\mu(g) \right|$$

$$\leq \int_G |h(g) - h(e)|\, p(g)\, d\mu(g)$$

$$\leq \varepsilon \int_G \varphi(g)\, d\mu(g)$$ ~ note that this integral is > 0.

Hence $|h(e)| \leq \varepsilon \quad \forall \varepsilon > 0$ and therefore $h(e) = 0$ $\square$

Proof of the uniqueness of the Haar measure.

We let μ be a left Haar measure and ν be a right Haar measure.

Let $f, g \in C_c(G)$ s.t. $\int f\, d\mu \neq 0$.

Then we can compute

$$\int_G f\, d\mu \int_G g\, d\nu = \int_G f\, d\mu \cdot \int_G g(y)\, d\nu(y)$$

$$= \int_G f\, d\mu \cdot \int_G g(yt)\, d\nu(y)$$ doesn't depend on t.

right invariant

$$= \int_G f(t) \left(\int_G g(yt)\, d\nu(y) \right) d\mu(t)$$

$$= \int_G \left(\int_G f(t) g(yt)\, d\mu(t) \right) d\nu(y)$$

Fubini

$$= \int_G \left(\int_G f(y^{-1}x) \cdot g(x) \, d\mu(x) \right) d\nu(y)$$

$x = yt$ + μ left invariant.

$$= \int_G \left(\int_G f(y^{-1}x) \, d\nu(y) \right) g(x) \, d\mu(x) \qquad (*)$$

Fubini

Note that we could use Fubini's theorem since the supports of f and g are compact.

Then we define $w_f : G \longrightarrow \mathbb{R}$ by

$$w_f(x) := \frac{1}{\int_G f \, d\mu} \int_G f(y^{-1}x) \, d\nu(y)$$

From $(*)$ we get:

$$\int g \, d\nu = \frac{1}{\int_G f \, d\mu} \int_G \left(\int_G f(y^{-1}x) \, d\nu(y) \right) g(x) \, d\mu(x)$$

$$= \int_G w_f(x) \cdot g(x) \, d\mu(x). \qquad (**)$$

The left hand side in $(**)$ is independent of f. Thus, for all $f_1, f_2 \in C_c(G)$

such that: $\int f_1 \, d\mu$, $\int f_2 \, d\mu \neq 0$ it holds.

$$0 = \int_G w_{f_1}(x) g(x) \, d\mu(x) - \int_G w_{f_2}(x) g(x) \, d\mu(x)$$

for all $g \in C_c(G)$.

// Verify that it is continuous

By Lemma 2.47 (ii). $w_{f_1} - w_{f_2} \equiv 0$.

In particular, there exists $C \in \mathbb{R}$ such that $w_f(e) = C$ for all $f \in C_c(G)$ s.t $\int_G f \, d\mu \neq 0$.

Thus

$C = w_f(e)$

$$\int_G f \, d\mu \cdot C = \int_G f \, d\mu \cdot \frac{1}{\int_G f \, d\mu} \int_G f(y^{-1}) \, d\nu(y)$$

$$= \int_G f(y^{-1}) \, d\nu(y)$$

$$= \int_G \check{f} \, d\nu.$$

for every $f \in C_c(G)$ s.t $\int_G f \, d\mu \neq 0$

Assume that μ' is another left Haar measure.

There is a right Haar measure ν s.t.

$$\int_G f \, d\nu = \int_G \check{f} \, d\mu' \qquad \text{Lemma 2.44}$$

$$\Rightarrow \quad \int_G \check{f} \, d\nu = \int_G f \, d\mu' \quad \forall f \in C_c(G).$$

Apply the above reasoning with this choice of ν. Then we get

$$c \int_G f \, d\mu = \int_G f \, d\mu' \quad \forall f \in C_c(G)$$

with $\int_G f \, d\mu \neq 0$

This is clearly sufficient (Exercise) to show that μ and μ' coincide up to a positive constant. $\square$

$\boxed{\text{Example 2.48}}$

① The Lebesgue measure on $(\mathbb{R}^n, +)$ is a left and right Haar measure.

② Let λ denote the Lebesgue measure on $\mathbb{R}$. then:

$$I(f) := \int_0^\infty f(x) \frac{d\lambda(x)}{x}, \quad f \in C_c(\mathbb{R}_{>0})$$

defines a left and right Haar functional on $(\mathbb{R}_{>0}, \cdot)$.

③ If G is a <u>discrete group</u> then the <u>counting measure</u> is a left and right Haar measure.

Exercise 2.49

Find a left Haar measure on

$$P = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x \in \mathbb{R}_{>0},\ y \in \mathbb{R} \right\}$$

Show that <u>it is not</u> a right Haar measure.

Might think that left Haar = right Haar ⇒ abelian: but this is not the case.

Exercise 2.50

Skip during lecture.

Let $dm(x) := \prod_{i,j} d\lambda(x_{i,j})$ be the Lebesgue measure on $M_{n,n}(\mathbb{R})$

Verify that $I(f) := \int_{GL(n,\mathbb{R})} f(x) \frac{dm(x)}{|\det x|^n}$

is a left and right Haar functional on $GL(n,\mathbb{R})$.

$\boxed{\text{Exercise 2.5.1}}$ or skip during lecture.

Let $N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$.

be the <u>Heisenberg group</u>. We can use the parametrization:

$$\mathbb{R}^3 \longrightarrow N$$

$$(x, y, z) \longmapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

to define: for any $f \in C_c(N)$

$$I(f) := \int_{\mathbb{R}^3} f(x, y, z)\, d\lambda(x)\, d\lambda(y)\, d\lambda(z)$$

Prove that in this way we obtain a left and right Haar functional.

We would like to understand <u>when a left Haar measure</u> is also <u>right invariant</u>.

$\boxed{\text{Notation}}$ Given a left Haar measure μ and $f \in C_c(G)$ we shall denote

$$\mu(f) := \int_G f\, d\mu.$$

We will use an analogous notation for right Haar measures.

Definition 2.52. [$\mathrm{Aut}(G)$]
We let $\mathrm{Aut}(G)$ be the group of continuous invertible automorphisms of G with continuous inverse.

Note that $\mathrm{Aut}(G)$ acts on $C_c(G)$ on the left via
$$(\alpha \cdot f)(x) := f(\alpha^{-1}(x)),$$
for $\alpha \in \mathrm{Aut}(G)$, $f \in C_c(G)$, $x \in X$.

Lemma 2.53
If μ is a left Haar measure on G then
$C_c(G) \ni f \longmapsto \mu(\alpha \cdot f)$
defines a left Haar functional.

Proof
We can compute for $g \in G$ and $x \in G$:
$$\alpha(\lambda(g)\, f)(x) = f(g^{-1}\alpha^{-1}(x)).$$

$$= f\left(\alpha^{-1}\left[\alpha(g)^{-1}x\right]\right)$$

$$= (\alpha\cdot f)\left(\alpha(g)^{-1}x\right)$$

$$= \lambda_{\alpha(g)}(\alpha\cdot f)(x)$$

Hence $\mu(\alpha\,\lambda(g).f) = \mu\left(\lambda_{\alpha(g)}(\alpha\cdot f)\right)$

μ left invariant $= \mu(\alpha\cdot f)$ $\square$

By the uniqueness up to positive multiplicative constants of left Haar measures, Thm 2.42, there exists a positive constant that we shall denote as $\mathrm{mod}_G(\alpha)$ s.t.

$$\mu(\alpha\cdot f) = \mathrm{mod}_G(\alpha)\,\mu(f) \quad (*)$$

$\forall f \in C_c(G)$.

$\boxed{\text{Lemma 2.54.}}$

The function $\mathrm{mod}_G : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is a homomorphism.

Proof

Start by noticing that the definition of $\mathrm{mod}_G(\alpha)$ is independent of the choice of the left Haar measure μ. (*).

Then we recall that $\forall \alpha, \beta \in \mathrm{Aut}(G)$ and $\forall f \in C_c(G)$ $(\alpha\beta)\cdot f = \alpha\cdot(\beta\cdot f)$ i.e, $\mathrm{Aut}(G)$ acts on the left on $C_c(G)$.

Therefore:

$$
\begin{aligned}
\mathrm{mod}_G(\alpha\beta)\,\mu(f) &= \mu((\alpha\beta)\cdot f) \\
&= \mu(\alpha\cdot(\beta\cdot f)) \\
\text{By def of } \mathrm{mod}_G(\alpha) \longrightarrow\ &= \mathrm{mod}_G(\alpha)\ \mu(\beta\cdot f) \\
\text{By def of } \mathrm{mod}_G(\beta) \longrightarrow\ &= \mathrm{mod}_G(\alpha)\,\mathrm{mod}_G(\beta)\,\mu(f)
\end{aligned}
$$

$\Rightarrow \mathrm{mod}_G(\alpha\beta) = \mathrm{mod}_G(\alpha)\cdot \mathrm{mod}_G(\beta)$. $\square$

We already considered a particular kind of automorphisms, the so-called inner automorphisms $\mathrm{int}_g : G \longrightarrow G$ for $g \in G$ defined by:

$$\mathrm{int}_g(x) := g x g^{-1} \qquad \forall x \in G.$$

$\boxed{\text{Note}}$: $$(\mathrm{int}_g \cdot f)(x) = f(g^{-1} x g)$$

Applying the construction above to <u>inner automorphism</u> we get:

$$\Delta_G : G \longrightarrow \mathbb{R}_{>0}.$$
$$g \longmapsto \mathrm{mod}_G(\mathrm{int}_g)$$

We call such a function Δ_G the <u>modular function</u> of G.

We note that:

$$\Delta_G(g)\mu(f) = \mu(\mathrm{int}_g \cdot f)$$
$$= \int_G f(g^{-1} x g)\, d\mu(x).$$

$$= \int_G f(xg)\, d\mu(x)$$

μ is left invariant

$$= \mu(\rho(g) f).$$

Definition of $\rho(g) f$

Hence $\mu(\rho(g) f) = \Delta_G(g)\, \mu(f)$ and:

the modular function captures the extent to which a left Haar measure fails to be right invariant.

Proposition 2.55

i) $\Delta_G : G \longrightarrow \mathbb{R}_{>0}$ is a continuous homomorphism.

ii) $\forall f \in C_c(G)$

$$\int_G f(x^{-1}) \Delta_G(x) \, d\mu(x) = \int_G f(x) \, d\mu(x)$$

——— o ———

Before proving Prop. 2.55 we discuss a few preliminary facts of independent interest.

Definition 2.56

A map $F : G \longrightarrow Y$ where (Y, d) is a metric space, is left uniformly continuous (resp. right uniformly cont.) if $\forall \varepsilon > 0$ there is a neigh. $U \ni e$ such that: $d(f(x), f(y)) < \varepsilon$ $\forall x, y$

with. $x^{-1} y \in U$ (resp. $xy^{-1} \in U$)

$\boxed{\text{Lemma 2.57.}}$

i) If $f \in C_c(G)$, then $f : G \longrightarrow \mathbb{C}$ is <u>left</u> and <u>right uniformly continuous</u>.

ii) If we endow $C_c(G)$ with the distance coming from the <u>sup-norm</u> $\|f\|_\infty := \sup_{x \in G} |f(x)|$

then $\forall f \in C_c(G)$ the maps

$$G \longrightarrow C_c(G)$$
$$g \longmapsto \lambda(g) f.$$

and

$$G \longrightarrow C_c(G)$$
$$g \longmapsto \rho(g) f.$$

are continuous.

<u>Proof</u>

We note that i) and ii) are actually equivalent statements (Exercise).

We prove the left continuity part in (i). The proof of the right continuity is completely analogous.

We find an open symmetric neigh. $U = U^{-1} \ni e$ with $\overline{U}$ compact and let $K := \operatorname{supp} f \cdot \overline{U}$

Claim: K is compact. (Exercise.)

use that $U = U^{-1}$

If x, y are such that $x^{-1} y \in U$ and $x \notin K$ then from $y \in xU$ we get from $xU \cap \operatorname{supp} f = \emptyset$ that $f(x) = f(y) = 0$. Similarly if $y \notin K$ using that $U = U^{-1}$.

By continuity of f, $\forall x \in K$ $\exists W_x \ni e$ neigh. of e with $W_x \subset U$ and
$$|f(x) - f(y)| < \frac{\varepsilon}{2} \quad \forall y \in K \cap (x \cdot W_x)$$

Let $V_x \ni e$ open with $V_x^{-1} = V_x$.

and $V_x^2 \subset W_x$. Since $K \subset \bigcup_{x \in K} x V_x$

by compactness there exist $x_1, \dots, x_n$ with

$$K \subset \bigcup_{i=1}^{n} x_i V_{x_i}$$

Let $V := \bigcap_{i=1}^{n} V_{x_i}$, this is going to be the neigh. doing the job.

If $x^{-1}y \in V$, $x \in K$, $y \in K$. then we let $1 \leq i \leq n$ be such that: $x \in x_i V_{x_i}$.

$x \in x_i V_{x_i}$, $V \subset V_{x_i}$ $\quad$ $V_{x_i}^2 \subset W_{x_i}$

Then $y \in xV \subset x_i V_{x_i}^2 \subset x_i W_{x_i}$

and hence $\quad < \frac{\varepsilon}{2}$, $x \in x_i V_{x_i}$

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)|$$
$$< \varepsilon$$

$\frac{\varepsilon}{2}$, $y \in x_i W_{x_i}$

If $x^{-1}y \in V \subset U$ and either $x \notin K$ or $y \notin K$ then we already noticed that $f(x) = f(y) = 0$ $\square$

Corollary 2.58

Let μ be a left Haar measure on G.

i) Assume that $F_1, F_2 : G \longrightarrow Y$ are continuous maps into a topological Hausdorff space such that:

$$F_1(g) = F_2(g) \quad \text{for } \mu\text{-a.e. } g \in G.$$

Then: $F_1(g) = F_2(g) \quad \forall g \in G$.

ii) Let $f \in C_c(G)$ with $f \geq 0$ and $f \neq 0$. Then: $\int_G f(x)\, d\mu(x) > 0$.

Proof

The statement follows directly from Lemma 2.47 (i) (Exercise) $\square$

Proof of Proposition 2.55

Proof of (i)

Since: $\Delta_G : G \longrightarrow \mathbb{R}_{>0}$ is a homomorphism it is sufficient to prove continuity at e.

Pick $f \in C_c(G)$ with $f \geq 0$ and $\mu(f) = 1$.

Then:

$$|(\Delta_G(g) - 1) \cdot 1| = \left| \int_G (f(\xi g) - f(\xi))\, d\mu(\xi) \right|$$

We assume that $g \in U = U^{-1}$ open neigh. of e s.t. $\overline{U}$ is compact.
Let $K := \operatorname{supp} f \cdot \overline{U}$ and note that:
if $\xi \notin K$ then: $f(\xi g) = f(\xi) = 0$

Hence.
$$\left| \int_G \left(f(\xi g) - f(\xi)\right) d\mu(\xi) \right| \leq \mu(K) \, \| \rho(g) f - f \|_\infty.$$

The continuity of Δ_G at $g = e$ then follows from the continuity at $g = e$ of:

$$\begin{aligned} G &\longrightarrow (C_c(G), \|\cdot\|_\infty) \\ g &\longmapsto \rho(g) f. \end{aligned}$$

that was obtained in Lemma 2.57 ii)

<u>Proof of (ii).</u>

Note that by (i), if $f \in C_c(G)$ then:
$f \cdot \Delta_G \in C_c(G)$ ↑ Δ_G is continuous by i)

We define. $I(f) := \int_G f(x^{-1})\, \Delta_G(x)\, d\mu(x)$

$\boxed{\text{Claim}}$: $I(f)$ is a <u>left Haar functional</u>.

Indeed:

$$I(\lambda(g)f) = \int_G f(g^{-1}x^{-1})\, \Delta_G(x)\, d\mu(x)$$

Δ_G homom

$$= \int_G f((xg)^{-1})\, \Delta_G(xg)\, d\mu(x)\, \Delta_G(g)^{-1}$$

Pog $\Delta_G(g)$ to cancel out $\delta(g)$

$$= \int_G f(x^{-1})\, \Delta_G(x)\, d\mu(x)\, \Delta_G(g)\, \Delta_G(g)^{-1}$$

$$= I(f).$$

Thm 2.4.2

Hence by <u>the uniqueness up to constants</u> of the left Haar measure there is $c > 0$. with

(*) $$\int_G f(x^{-1})\, \Delta_G(x)\, d\mu(x) = c \int_G f(y)\, d\mu(y).$$

<u>Claim : $c = 1$</u>

Write $\int_G f(y)\, d\mu(y) = \int_G \left(f(y)\, \Delta_G(y^{-1})\right) \Delta_G(y)\, d\mu(y)$

and set $F(y) := f(y^{-1}) \Delta_G(y)$.

Then: $\int_G f(y)\, d\mu(y) = \int_G F(y^{-1}) \Delta_G(y)\, d\mu(y)$

$\overset{(*)}{=} c \cdot \int_G F(y)\, d\mu(y) = c \int_G f(y^{-1}) \Delta_G(y)\, d\mu(y)$

$\overset{(*)}{=} c^2 \int_G f(z)\, d\mu(z)$ def of F

Thus $\int_G f(y)\, d\mu(y) = c^2 \int_G f(z)\, d\mu(z)$

and choosing f with $\mu(f) \neq 0$ we get $c = 1$, since $c > 0$ $\square$

Definition 2.59

A locally compact Hausdorff group G is said to be <u>unimodular</u> if $\Delta_G(g) = 1$, $\forall g \in G$. Equivalently, <u>every left Haar measure is a right Haar measure.</u>

Example 2.60

$(\mathbb{R}^n, +)$, $(\mathbb{R}_{>0}, \cdot)$, <u>discrete groups.</u>

$GL(n,\mathbb{R})$, the Heisenberg group are unimodular.

On the other hand $P = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \quad \begin{matrix} x > 0 \\ y \in \mathbb{R} \end{matrix} \right\}$.

is not unimodular.

$\mu(f) = \mu(\check{f})$.

$\forall f \in C_c(G)$

Corollary 2.61

The Haar measure of a group is inverse invariant if and only if the group is unimodular

$\mu(A) = \mu(A^{-1})$

Proof

We have seen that if μ is left Haar then $f \longmapsto \mu(\check{f})$ defines a right Haar functional $\square$

Proposition 2.62

A locally compact Hausdorff group G is compact iff it has finite Haar measure.

Proof

μ left Haar measure on G.

G compact $\Longrightarrow \mu(G) < \infty$ by regularity

Assume now $\mu(G) < \infty$. By regularity there is $K \subset G$ compact with
$$\mu(K) > \frac{1}{2}\mu(G).$$

Let $g \in G$ then $\mu(gK) = \mu(K)$ hence $\mu(gK) + \mu(K) > \mu(G)$.

$$\Rightarrow \quad gK \cap K \neq \emptyset$$
$$\Rightarrow \quad g \in KK^{-1}$$
$$\Rightarrow \quad G = KK^{-1}$$
$\Rightarrow$ G compact $\square$

$\boxed{\text{Example 2.6.3}}$

- Any locally compact Hausdorff <u>abelian</u> group is <u>unimodular</u> (trivial).
- Any <u>compact and Hausdorff group</u> is <u>unimodular</u> because there are no compact non-trivial subgroups in $(\mathbb{R}_{>0}, \cdot)$

2.5.2. Homogeneous spaces

Homogeneous spaces of locally compact Hausdorff groups and of Lie groups in particular play a very important role in geometry, topology, and dynamical systems.

We are going to discuss the basics and the existence of invariant measures.

Let G be a top. group, $H < G$ a subgroup, and G/H the set

$$\{x \cdot H : x \in G\}$$

of right H-cosets. Let $p : G \longrightarrow G/H$ be the canonical projection.

We endow G/H with the quotient topology, that is to say $U \subset G/H$ is open if and only if $p^{-1}(U)$ is open in G.

Proposition 2.64

Let $H < G$ be a subgroup of a top. group G.

1) The projection $p : G \longrightarrow G/_H$ is an <u>open map</u>.
2) The <u>action</u> $G \times G/_H \longrightarrow G/_H$ is <u>continuous</u>.
3) $G/_H$ is <u>Hausdorff</u> iff <u>H is closed</u>.
4) G <u>locally compact</u> $\Longrightarrow$ $G/_H$ <u>loc. compact</u>.
5) If G is loc. compact and $H < G$ is closed. then $\forall$ $C \subset G/_H$ compact, there is $K \subset G$ compact s.t. $p(K) = C$.

Proof

The proofs of 1) and 2) are left as an Exercise.

<u>Proof of 3</u>

$G/_H$ Hausdorff $\Longrightarrow$ points are closed.

$\Longrightarrow$ H is closed.

Assume H is closed. If $xH \neq yH$ then $xHy^{-1} \not\ni e$. and since xHy^{-1} is closed. there is $V \ni e$ open with $V^{-1}V \cap xHy^{-1} = \emptyset$

Equivalently $Vy \cap VxH = \emptyset$ and hence $VyH \cap VxH = \emptyset$

by 1).

Thus $p(Vy)$ and $p(Vx)$ are open neighs of yH and xH respectively with $p(Vy) \cap p(Vx) = \emptyset$

4). Let $p(x) \in G/H$. G is loc. compact hence there is $x \in U \subset C \subset G$ with U open and C compact,

Then $p(x) \in p(U) \subseteq p(C)$ and $p(U)$ is open by 1) and $p(C)$ is compact by continuity of p

5) Let $e \in U \subset L$ with U open and L compact.

Then $\bigcup p(xU)$ is an open

covering $\underset{x \in G}{}$ of C. Hence there are $x_1, \dots, x_n$ s.t.:

$$C \subset \bigcup_{i=1}^{n} p(x_i U).$$

In particular $C \subset \bigcup_{i=1}^{n} p(x_i L)$

Since H is closed, G/H is Hausdorff by 3). Therefore C being compact it is closed.

Therefore $K := p^{-1}(C) \cap (\bigcup_{i=1}^{n} x_i L)$ is compact since it is a closed subset of the compact set $\bigcup_{i=1}^{n} x_i L$

Moreover $p(K) = C$ $\square$

Connected with respect to the lecture!

In order to produce examples we note the following. If G acts <u>transitively</u> on a space X then there is an <u>isomorphism of G spaces</u> $G/G_x \longrightarrow X$ where $G_x = \mathrm{Stab}_G(x)$ is the <u>stabilizer</u> of x. The isomorphism is given by the map. $\quad gG_x \longmapsto gx.$

If X is a topological space and the action of G on X is continuous, then the G-map above is also continuous.

If G is <u>locally compact, second countable and Hausdorff</u> and X is <u>locally compact Hausdorff</u> then the bijection is a <u>homeomorphism</u>.

we shall call X a homogeneous space in this case.

$\boxed{\text{Example 2.65}}$

Consider the action of $O(n+1, \mathbb{R})$ on $S^n \subseteq \mathbb{R}^{n+1}$. As we already noticed

$${}^t g g = \mathrm{Id} \implies \|gv\| = \|v\| \quad \forall v \in \mathbb{R}^{n+1}$$

Hence S^{n+1} is preserved by $O(n+1, \mathbb{R})$,

Moreover the action is transitive, i.e,

$$O(n+1, \mathbb{R})\, e_{n+1} = S^n.$$

In fact, even the action of $SO(n+1, \mathbb{R})$ is transitive on S^n.

The stabilizer of $e_{n+1} \in S^n$ is:

$$SO(n+1,\mathbb{R})_{e_{n+1}} = \{ g \in SO(n+1,\mathbb{R}) : g e_{n+1} = e_{n+1} \}$$

$$\cong \left\{ \left(\begin{array}{c|c} h & 0 \\ \hline 0 & 1 \end{array} \right) : h \in SO(n,\mathbb{R}) \right\} \cong SO(n,\mathbb{R}),$$

Therefore $$S^n \cong SO(n+1,\mathbb{R}) / SO(n,\mathbb{R})$$

↳ look at Remark 2.71

More in general, in $\mathbb{R}^n$ with the standard scalar product we let $1 \le k \le n$ and consider

$$GO_k := \{ (v_1, \dots, v_k) \in (\mathbb{R}^n)^k : v_1, \dots v_k \text{ is an orthonormal set} \}$$

Then $O(n,\mathbb{R})$ acts transitively on GO_k by

$$(g, (v_1, \dots, v_k)) \longmapsto (g v_1, \dots, g v_k)$$

$g \in O(n,\mathbb{R})$, $(v_1, \dots, v_k) \in GO_k$.

Infact if $1 \le k \le n-1$ then $SO(n,\mathbb{R})$

acts transitively on G_{2_k}.

The stabilizer of $(e_1, \dots, e_k)$ is:

$$O(n,\mathbb{R})_{(e_1,\dots,e_k)} = \left\{ \begin{pmatrix} Id_k & 0 \\ 0 & h \end{pmatrix} : h \in O(n-k,\mathbb{R}) \right\}$$

$$\simeq O(n-k,\mathbb{R})$$

Hence $\quad G_{2_k} \simeq O(n,\mathbb{R}) / O(n-k,\mathbb{R})$

In order to discuss the next example we need:

Definition 2.66

A lattice Γ in a locally compact Hausdorff group G is a subgroup with the following properties:

1) Γ is discrete.

2) there exists on G/Γ a finite G-invariant regular Borel measure.

$\boxed{\text{Example 2.67}}$ ↳ look at Exercise 2.72 below!

Let $\mathcal{R}$ be the set of lattices in $\mathbb{R}^n$.

Then: (Exercise.) $\Gamma \in \mathcal{R}$ iff there exists a basis $f_1, \dots, f_n$ of $\mathbb{R}^n$ such that $\Gamma = \mathbb{Z} f_1 + \dots + \mathbb{Z} f_n$.

Note that $GL(n,\mathbb{R})$ acts transitively on $\mathcal{R}$.

If we let $\Gamma_0 = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_n$ then

$$GL(n,\mathbb{R})_{\Gamma_0} = GL(n,\mathbb{Z}) = \{ A \in M_{n,n}(\mathbb{Z}) : \det A = \pm 1 \}$$

Therefore.
$$\mathcal{R} \simeq GL(n,\mathbb{R}) / GL(n,\mathbb{Z})$$

The question that we would like to address now is:

$\boxed{\text{Question:}}$ Consider a locally compact.

Hausdorff group G and a closed subgroup $H < G$ so that $G/_H$ is locally compact Hausdorff and the G action is continuous. Prop. 2.64

When does there exist a G-invariant positive functional on $C_c(G/_H)$?

Equivalently, when does there exist a <u>G-invariant</u> <u>positive regular Borel</u> measure on $G/_H$?

A complete answer is given by the following:

$\boxed{\text{Theorem 2.68}}$ [Weil formula]

Let G be <u>locally compact Hausdorff</u> with <u>left Haar measure μ_G</u> and $H < G$ be a <u>closed subgroup</u> with left Haar measure μ_H.

Then there is a G-invariant positive regular Borel measure on $G/_H$ iff

$$\Delta_G|_H = \Delta_H.$$

In this case, said regular Borel measure is unique up to positive scalar multiples and there is a unique choice $\mu_{G/H}$ such that Weil's formula holds.

$$\int_G f(g)\, d\mu(g) = \int_{G/H} \int_H f(gh)\, d\mu_H(h)\, d\mu_{G/H}(gH)$$

$\forall f \in C_c(G)$.

It is very instructive to think about the case where $G = \mathbb{R}^2$ and $H = \mathbb{R} \times \{0\} < \mathbb{R}^2$.

Let's start with some preliminary considerations.

Let $f \in C_c(G)$, and $g \in G$.
The function $H \ni h \longmapsto f(gh)$ is in $C_c(H)$.
Hence $T_H f(g) := \int_H f(gh)\, d\mu_H(h)$ is well-defined.

By Lemma 2.57 the map $g \longmapsto T_H f(g)$ is continuous.

By (left) H-invariance of μ_H we also have.

$$T_H f(gu) = T_H f(g) \quad \forall g \in G \quad \forall u \in H.$$

Hence $T_H f$ can be considered as a function on G/H

It is straightforward to check that if $p: G \longrightarrow G/H$ denotes the projection map. then $\operatorname{supp}(T_H f) \subseteq p(\operatorname{supp} f)$ and thus $T_H f \in C_c(G/H)$.

⇝ so the Weil formula makes sense!

Then we have the following.

$\boxed{\text{Lemma 2.69}}$ (Skipped during the lecture).

The map $T_H : C_c(G) \longrightarrow C_c(G/H)$ is surjective.

In addition, if $F \in C_c(G/H)$ is ≥ 0, we can find $f \in C_c(G)$ $f \geq 0$ such that $T_H f = F$

Proof

We give a sketch of the proof.

Let $F \in C_c(G/H)$. By Prop 2.64 (5) we can find $K \subset G$ compact such that $p(K) = \operatorname{supp} f$

By Urysohn's Lemma there exists $\eta \in C_c(G)$ such that $0 \leq \eta \leq 1$ and $\eta|_K \equiv 1$.

We define $f : G \longrightarrow \mathbb{C}$ by

$$f(g) := \begin{cases} \dfrac{F \circ p(g) \cdot \eta(g)}{T_H \eta(g)} & \text{if } T_H \eta(g) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Claim</u>: f is continuous.

The claim is clear on the open set
$$U_1 := \{ g \in G : T_H \eta(g) \neq 0 \}$$

and on the open set (check that it is indeed open)
$$U_2 := G \setminus KH = p^{-1}(G/H \setminus \operatorname{supp} F)$$
since it vanishes there.

Note that if $g \notin U_1$ then $\eta(gh) = 0$ for μ_H-a.e. $h \in H$. By continuity, $\eta(gh) = 0$ for every $h \in H$.

Then (Exercise) $g \notin KH$.

Thus $U_1 \cup U_2 = G$. Hence f <u>is continuous.</u>

Since $\operatorname{supp} f \subset \operatorname{supp} \eta$, $f \in C_c(G)$

<u>Claim</u>: $T_H f = F$

If $g \in U_1$.

$$T_H f(g) = \int_H \frac{(F \circ p)(gh)\, \eta(gh)}{T_H \eta(gh)} \, d\mu_H(h)$$

$$= \frac{F \circ p(g)}{T_H \eta(g)} \int_H \eta(gh) \, d\mu_H(h)$$

$$= F \circ p(g)$$

If $g \in U_2$ then on the other hand: $f(gh) = 0$

$\forall h \in H$. hence. $T_H f(g) = 0 = F \circ p(g)$ $\square$

Proof of Theorem 2.68 (Skipped during the lecture).

Assume that there is a G-invariant regular Borel measure $\mu_{G/H}$ on G/H.

Then for $f \in C_c(G)$ setting

$$I(f) := \int_{G/H} (T_H f)(\dot{g})\, d\mu_{G/H}(\dot{g})$$

we obtain a left Haar functional on G.

Indeed:

$$I(\lambda(g) f) = \int_{G/H} (T_H(\lambda(g) f))(\dot{g})\, d\mu_{G/H}(\dot{g})$$

left and right translations commute.

$$= \int_{G/H} (\lambda(g) T_H f)(\dot{g})\, d\mu_{G/H}(\dot{g})$$

$$= \int_{G/H} (T_H f)(\dot{g})\, d\mu_{G/H}(\dot{g})$$

G-left invariance of $\mu_{G/H}$.

In particular.

$$I(\rho(t)f) = \Delta_G(t) \cdot I(f) \qquad \forall\ t \in H.$$

On the other hand.

$$I(\rho(t)f) = \int_{G/H} T_H(\rho(t)f)(\dot{g})\, d\mu_{G/H}(\dot{g})$$

and:

$$T_H(\rho(t)f) = \int_H f(ght)\, d\mu_H(h)$$

$$= \Delta_H(t)\, T_H f(g).$$

Thus: $\Delta_G|_H = \Delta_H$

Let us assume now that: $\Delta_G|_H = \Delta_H$

Claim: $\forall\ f_1, f_2 \in C_c(G)$ it holds.

$$\int_G f_1(g) T_H f_2(g)\, d\mu(g) = \int_G f_2(g) T_H f_1(g)\, d\mu(g)$$

Indeed.

$$\int_G f_1(g) \int_H f_2(gh)\, d\mu_H(h)\, d\mu_G(g)$$

$$= \int_H d\mu_H(h) \underbrace{\int_G f_1(g) f_2(gh)\, d\mu_G(g)}_{\Delta_G(h) \int_G f_1(gh^{-1}) f_2(g)\, d\mu_G(g)}$$

$$= \int_G f_2(g) \int_H f_1(gh^{-1})\, \underbrace{\Delta_G(h)}_{\Delta_H(h)}\, d\mu_H(h)\, d\mu_G(g)$$

Prop 2.55 ii)

$$= \int_G f_2(g) \int_H f_1(gh)\, d\mu_H(h)\, d\mu_G(g)$$

$$= \int_G f_2(g)\, T_H f_1(g)\, d\mu_G(g)$$

Now for any $F \in C_c(G/H)$ we can choose any $f \in C_c(G)$ with $T_H f = F$ wa: Lemma 2.69

We set: $\mathcal{Y}(F) := \int_G f(g)\, d\mu_G(g)$.

<u>Claim</u>: $\mathcal{Y}$ is well-defined

In order to establish the claim we need to show that: if $\varphi \in C_c(G)$ is such that: $T_H \varphi = 0$ then: $\int_G \varphi(g)\, d\mu_G(g) = 0$

Choose: $\psi \in C_c(G)$ such that:

$$T_H \psi(\dot g) = 1 \qquad \forall \dot g \in p(\operatorname{supp} \varphi)$$

and compute:

$$\int_G \varphi(g)\, d\mu_G(g) = \int_G \varphi(g)\, T_H \psi(g)\, d\mu_G(g)$$

$$= \int_G \psi(g)\, \underbrace{T_H \varphi(g)}_{=0}\, d\mu_G(g) = 0$$

Thus $\mathcal{Y} : C_c(G/H) \longrightarrow \mathbb{C}$ is well-defined.

The proof that it is a positive invariant functional is left as an Exercise.

$\square$

We end this section by discussing a <u>necessary condition</u> for the <u>existence of a lattice</u>.

Proposition 2.70
Let G be a <u>locally compact Hausdorff group</u> that <u>admits a lattice</u> $\Gamma < G$. Then <u>G is unimodular</u>.

Proof
By definition of lattice there is a <u>finite</u> <u>G-invariant</u> measure $\mu_{G/\Gamma}$ on G/Γ.

By Thm 2.68 $\Delta_G|_\Gamma = \Delta_\Gamma$. On the other hand Γ is <u>discrete</u> and hence <u>unimodular</u>. Hence $\Gamma < \operatorname{Ker} \Delta_G$.

Therefore Δ_G descends to a Δ_G map $\Delta : G/\Gamma \longrightarrow \mathbb{R}_{>0}$, that is a map such that for $h \in G$ and $x \in G/\Gamma$.

$$\Delta(hx) = \Delta_G(h)\,\Delta(x)$$

The push-forward via Δ of the finite

G-invariant measure on G/Γ, $\Delta_* \mu_{G/\Gamma}$ is a finite $\Delta_G(G)$ invariant measure on $\mathbb{R}_{>0}$.

This is impossible unless $\Delta_G(G) = \{1\}$
(Check this!) $\square$

In general $f_* \mu(A) := \mu(f^{-1}(A))$, is the definition of the push-forward of μ via f.

Remark 2.71

In the setting that we discussed before Example 2.65. we note the following:

i). The stabilizer of a point $x \in X$ in general depends on x as a subgroup of G.

Consider for instance the action of $SO(3)$ on $S^2 = \{ \|v\| = 1 \} \subseteq \mathbb{R}^3$ that we already discussed.

Then: $SO(3)_{e_1} = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & h \end{array} \right) : h \in SO(2) \right\}$

on the other hand:

$$SO(3)_{e_3} = \left\{ \left(\begin{array}{c|c} h & 0 \\ \hline 0 & 1 \end{array} \right) : h \in SO(2) \right\}$$

So $SO(3)_{e_1} \neq SO(3)_{e_3}$ as subgroups of $SO(3)$. However <u>they are both isomorphic</u> to $SO(2)$.

ii) This is not a coincidence, if G acts transitively on X, then G_x and $G_{x'}$ <u>are conjugated subgroups</u> of G.

$G_x = \{ g \in G : gx = x \}$

$G_{x'} = \{ g' \in G : g'x' = x' \}$

G-action transitive

Let $h \in G$ be such that $hx = x'$

If $g' \in G_{x'}$ then

$g'hx = g'x' = x' = hx. \implies h^{-1}g'hx = x.$

$\implies h^{-1}g'h \in G_x.$

$\implies h^{-1}G_{x'}h \subseteq G_x.$

On the other hand, if $g \in G_x$

then $gx = x \implies hgx = hx = x'$

$\implies hgh^{-1}hx = x'$

$\implies hgh^{-1}x' = x' \implies hgh^{-1} \in G_{x'}$

Hence $hG_xh^{-1} \subseteq G_{x'}$ which is enough to show that $hG_xh^{-1} = G_{x'}$

(iv) The example above also shows that the <u>stabilizer need not be a normal subgroup in general</u>. For instance $SO(3)_{e_1}$ is not normal in $SO(3)$.

(v) Therefore G/G_x <u>does not</u> come with a naturally induced group structure in general.

In this specific example $SO(3)/SO(3)_{e_1} \approx S^2$.

We note that S^2 <u>does not</u> admit <u>any topological group</u> structure. (although this is not obvious at all).

r) In general: it is not true that:

$$G/_{G_x} \times G_x \approx G$$ as topological spaces.

Indeed in the above example:

$SO(3) \approx \mathbb{RP}^3$, $SO(3)_{e_1} \approx S^1$.

$$SO(3)/SO(3)_{e_1} \approx S^2$$

However, $\mathbb{RP}^3 \not\approx S^2 \times S^1$

$\Rightarrow$ for instance $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/_2\mathbb{Z}$ $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$

$\boxed{\text{Exercise: 2.72}}$

In Definition 2.66 the requirement is that the invariant measure on $G/_\Gamma$ is finite.

We saw in Proposition 2.62 that a loc. compact Hausdorff group has finite Haar measure iff it is compact.

c) Understand why the proof doesn't work for $G/_\Gamma$ to show that if it has finite invariant measure then it is compact.

d) Find an example of loc. compact Hausdorff group G with a lattice Γ such that $G/_\Gamma$ is not compact.

— o —

We discuss an interesting application of Haar measures:

Proposition 2.73

Any compact subgroup of $GL(n,\mathbb{R})$ is conjugate to a subgroup of $O(n,\mathbb{R})$

Proof

Let $(\ ,\ )$ be any positive definite scalar product on $\mathbb{R}^n$ and μ be a Haar measure on $K < GL(n,\mathbb{R})$.

We let $\langle\ ,\ \rangle$ be defined by

$$\langle v,w\rangle := \int_K (gv,gw)\,d\mu(g)$$

We claim that: $\langle\ ,\ \rangle$ is a positive definite scalar product on $\mathbb{R}^n$ and that:
$\forall g \in K$ it holds $\langle gv,gw\rangle = \langle v,w\rangle$, $\forall v,w \in \mathbb{R}^n$, i.e. $\langle\ ,\ \rangle$ is K-invariant.

Note : it is important that K is compact, otherwise the integral might not converge.

Symmetry, bilinearity, as well as positive definiteness are elementary and left as an Exercise.

We check invariance.

$$\langle hv, hw \rangle \overset{\text{def of } \langle , \rangle}{=} \int_K (gh.v, ghw)\, d\mu(g)$$

$$= \int_K (gv, gw)\, d\mu(g)$$

μ is both right and left invariant since K is compact and hence unimodular.

$$= \langle v, w \rangle \quad \forall v, w \in \mathbb{R}^n$$

def of $\langle , \rangle$ $\quad \forall h \in K.$

The conclusion that K is conjugate to a subgroup of $O(n,\mathbb{R})$ follows. Indeed, we let A be a positive definite symmetric matrix representing $\langle , \rangle$ with respect to the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. i.e.,

$\langle v, w \rangle = {}^t v A w \quad \forall v, w \in \mathbb{R}^n.$

Let B be the unique positive definite symmetric square root of A, i.e., $B = {}^t B$, $B^2 = A$.

If $g \in K$, then we claim that $B^{-1} g B \in O(n)$.

Indeed: $(B^{-1} g B)^t (B^{-1} g B) = B^{-1} g^t B B^t g^t (B^{-1})$ See below

$\overset{!}{=} B^{-1} g B^{2} {}^t g B^{-1}$

${}^t(B^{-1}) = B^{-1}$
$B = {}^t B$

$\wr$

$= B^{-1} B^2 B^{-1} = \mathrm{Id}.$

$g B^2 {}^t g$
$\parallel$
$g A {}^t g = A = B^2.$

Therefore $B^{-1} K B < O(n)$, as claimed $\square$

<u>Note after the lecture</u>: this doesn't quite work in this way. We need to compute instead:

$${}^t(B g B^{-1}) (B g B^{-1}) = {}^t(B^{-1})^t g \, {}^t B B g B^{-1}$$
$$= B^{-1} {}^t g B^2 g B^{-1}$$
$$= B^{-1} {}^t g A g B^{-1}$$
$$= B^{-1} B^2 B^{-1} = \mathrm{Id}.$$

Hence $B g B^{-1} \in O(n)$.

Note after the lecture: for many of the "topological" statements in this chapter it is very helpful to think about the proofs in the case where the groups are abelian (or even $\mathbb{R}^n$).

The proofs in the general cases are often a slightly more technical version of those working in $\mathbb{R}^n$.